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Concrete Quantum Logics with Covering Properties

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Let L be a concrete (=set-representable) quantum logic. Let n be a natural number (or, more generally, a cardinal). We say that L admits intrinsic coverings of the order n, and write $L\in\mathscr{C}_n$, if for any pair $A, B\in L$ we can find a collection $\{C_i: i\in I\}$, where card $I<n$ and $C_i\in L$ for any $i\in I$, such that $A\cap B=\bigcup_{i\in I} C_i$. Thus, in a certain sense, if $L\in\mathscr{C}_n$, then "the rate of noncompatibility" of an arbitrary pair $A, B\in L$ is less than a given number n. In this paper we first consider general and combinatorial properties of logics of $\mathscr{C}_n$ and exhibit typical examples. In particular, for a given n we construct examples of $L\in\mathscr{C}_{n+1}\setminus\mathscr{C}_n$. Further, we discuss the relation of the classes $\mathscr{C}_n$ to other classes of logics important within the quantum theories (e.g., we discover the interesting relation to the class of logics which have an abundance of Jauch–Piron states). We then consider conditions on which a class of concrete logics reduce to Boolean algebras. We conclude with some open questions.

1. PRELIMINARIES

Among the quantum logics, whose significance within the axiomatics of quantum theories has been advocated in e.g., Birkhoff and von Neumann (1936), Varadarajan (1968), Jauch (1968), Piron (1976), Gudder (1979), and Pták and Pulmannová (1991), a special conceptual role is taken by concrete logics. A concrete (quantum) logic L is one which admits a set representation. In this paper we shall exclusively deal with concrete logics.

It is known (e.g., Gudder, 1979; Pták and Wright, 1985) that two sets $A, B\in L$ form a compatible pair in L if and only if $A\cap B\in L$. Obviously, if L should model a "genuinely quantum experiment" it has to contain noncompatible pairs (and, therefore, it cannot be a Boolean algebra). In this paper we consider those concrete logics where the relation of noncompatibility can be "approximated" by elements of L. As we shall see, apart

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from a potential application in quantum theories, these logics also enjoy interesting combinatorial properties.

Let us now recall the main notion we shall deal with in the sequel.

Definition 1.1. A *concrete logic* is a pair (X, L), where L is such a collection of subsets of X which fulfills the following properties:

(1) $\emptyset \in L$.
(2) $A^c = X \setminus A \in L$ whenever $A \in L$.
(3) $A \cup B \in L$ whenever $A, B \in L$ with $A \cap B = \emptyset$.

Thus, concrete logics are (nonvoid) collections of subsets of a set which contain the empty set and which are closed under the formation of the complements and of the disjoint (finite) unions. Observe also that if $A, B \in L$ and $A \subset B$, then $B \setminus A = (A \cup B^c)^c \in L$.

2. CONCRETE LOGICS WITH "COVERING PROPERTIES"

A concrete logic (X, L) is a Boolean algebra if and only if $A \cap B \in L$ for any $A, B \in L$. Thus, for a general logic, it is natural to introduce a classification of logics expressed in terms of how many of their elements are needed for the covering of intersections. This is done in the following definition. [Also, the definition has certain bearing on the physically significant notion of compatibility (respectively noncompatibility), as we indicated in the introduction.

Definition 2.1. Let α be a cardinal. Then $\mathscr{C}_\alpha$ denotes the class of concrete logics which are determined by the following property: If $A, B \in L$ with $A \neq B$, then there is a collection $\{C_i : i \in I\}$, where card $I < \alpha$ and $C_i \in L$ for any $i \in I$, such that $A \cap B = \bigcup_{i \in I} C_i$.

Let us first consider the relation between the classes $\mathscr{C}_\alpha$. Let us start with a simple observation. [Note that in our classification, the class $\mathscr{C}_2$ is exactly the class of all (concrete) Boolean algebras.]

Proposition 2.2. The following relation holds: $\emptyset \neq \mathscr{C}_0 \subsetneq \mathscr{C}_1 \subsetneq \mathscr{C}_2 \subsetneq \mathscr{C}_3$.

Proof. The inclusion $\mathscr{C}_0 \subset \mathscr{C}_1 \subset \mathscr{C}_2 \subset \mathscr{C}_3$ is obvious. Moreover, $\mathscr{C}_0 = \{(\emptyset, \{\emptyset\})\} \neq \emptyset$. The class $\mathscr{C}_1$ is the class of all concrete logics (X, L) such that $L = \{\emptyset, X\}$. Hence, $\mathscr{C}_0 \neq \mathscr{C}_1$. The class $\mathscr{C}_2$ is the class of concrete Boolean algebras. Hence, $\mathscr{C}_1 \neq \mathscr{C}_2$. It remains to prove that $\mathscr{C}_2 \neq \mathscr{C}_3$. Put $X = [0, 1]$ and let L be the set of all Borel subsets of the real interval $[0, 1]$ such that their Lebesgue measure is a rational number. Then $(X, L) \in \mathscr{C}_3 \setminus \mathscr{C}_2$. Indeed, every Borel subset of $[0, 1]$ is a union of two Borel sets with a rational Lebesgue measure and, on the other hand, there are Borel subsets of [0, 1] with a rational Lebesgue measure such that the Lebesgue measure of their intersection is not rational. ■

Proposition 2.3. Let n be a natural number with $n \geq 3$. Then $\mathscr{C}_n \subsetneqq \mathscr{C}_{n+1}$.

Proof. The inclusion $\mathscr{C}_n \subset \mathscr{C}_{n+1}$ is obvious. Let us construct a concrete logic $(X, L) \in \mathscr{C}_{n+1} \backslash \mathscr{C}_n$. Put

$$X_0 = \{a, b, c, d\}$$

$$L_0 = \{\varnothing, \{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, X_0\}$$

$$Y = \{0, 1, 2, \ldots, n\}$$

$$K_1 = \{\varnothing\} \cup \{\{0, y\} : y \in Y \backslash \{0\}\}$$

$$K_2 = \{Y \backslash B : B \in K_1\}$$

Then (X_0, L_0), $(Y, K_1 \cup K_2)$ are concrete logics. Let us inductively define a sequence of concrete logics (X_k, L_k), $k \geq 1$. First, for every $A \subset X_{k-1} \times Y$, let us write

$$P_x(A) = \{y \in Y : (x, y) \in A\} \qquad \text{for every } x \in X_{k-1}$$

$$P_1(A) = \{x \in X_{k-1} : P_x(A) \in K_1\}$$

$$P_2(A) = \{x \in X_{k-1} : P_x(A) \in K_2\}$$

Now, put

$$X_k = X_{k-1} \times Y$$

$$L_k = \{A \subset X_k : P_1(A), P_2(A) \in L_{k-1}, P_1(A) = P_2(A)^c\}$$

We shall prove (by induction) that (X_k, L_k) is a concrete logic. Indeed, $\varnothing = X_{k-1} \times \varnothing \in L_k$. For any $A \in L_k$ we have $P_x(A^c) = Y \backslash P_x(A)$. Hence, $P_1(A^c) = P_2(A)$, $P_2(A^c) = P_1(A)$ and therefore $A^c \in L_k$. Finally, suppose that $A, B \in L_k$ with $A \cap B = \varnothing$. Then $P_x(A) \cap P_x(B) = \varnothing$ for every $x \in X_{k-1}$ and therefore $P_2(A) \cap P_2(B) = \varnothing$ and $P_2(A \cup B) = P_2(A) \cup P_2(B) \in L_{k-1}$. On the other hand, $P_x(A \cup B) = P_x(A) \cup P_x(B) \in K_1 \cup K_2$ for every $x \in X_{k-1}$. We infer that $P_1(A \cup B) = X_{k-1} \backslash P_2(A \cup B)$.

Now, let us define

$$X = X_0 \times \prod_{i=1}^{\infty} Y$$

$$L = \bigcup_{k=0}^{\infty} \left\{A_k \times \prod_{i=k+1}^{\infty} Y : A_k \in L_k\right\}$$

It is easy to see that (X, L) is a concrete logic. It remains to be proved that $(X, L) \in \mathscr{C}_{n+1} \backslash \mathscr{C}_n$.

First, suppose that $A, B \in L$. Then there is a natural number k such that $A = A_k \times \Pi_{i=k+1}^{\infty} Y$, $B = B_k \times \Pi_{i=k+1}^{\infty} Y$, and $A_k, B_k \in L_k$. Since

$$A \cap B = (A_k \cap B_k) \times \prod_{i=k+1}^{\infty} Y$$
$$= \bigcup_{y \in Y \setminus \{0\}} \left((A_k \cap B_k) \times \{0, y\} \times \prod_{i=k+2}^{\infty} Y \right)$$

and $(A_k \cap B_k) \times \{0, y\} \in L_{k+1}$ for every $y \in Y \setminus \{0\}$, we have $(X, L) \in \mathscr{C}_{n+1}$. Finally, let us suppose that the set

$$\{a\} \times \prod_{k=1}^{\infty} Y = \left(\{a, b\} \times \prod_{k=1}^{\infty} Y \right) \cap \left(\{a, d\} \times \prod_{k=1}^{\infty} Y \right)$$

can be expressed as a union of m elements of L, where $m < n$. Let us seek a contradiction. There is a natural number $k \geq 1$ such that

$$\{a\} \times \prod_{k=1}^{\infty} Y = \bigcup_{j=1}^{m} \left(A_{k,j} \times \prod_{i=k+1}^{\infty} Y \right)$$

for some $A_{k,j} \in L_k$, $j \in \{1, \ldots, m\}$. For every $x \in \{a\} \times \Pi_{i=1}^{k-1} Y$ we have

$$Y = P_x \left(\{a\} \times \prod_{k=1}^{k} Y \right) = \bigcup_{j=1}^{m} P_x(A_{k,j})$$

Hence, $P_x(A_{k,j}) \in K_2$ and therefore $x \in P_2(A_{k,j})$ for some $j \in \{1, \ldots, m\}$. Thus,

$$\{a\} \times \prod_{k=1}^{\infty} Y = \bigcup_{j=1}^{m} \left(P_2(A_{k,j}) \times \prod_{i=k}^{\infty} Y \right)$$

where $P_2(A_{k,j}) \in L_{k-1}$ for every $j \in \{1, \ldots, m\}$. Proceeding by induction, we obtain

$$\{a\} \times \prod_{k=1}^{\infty} Y = \bigcup_{j=1}^{m} \left(A_{0,j} \times \prod_{i=1}^{\infty} Y \right)$$

for some $A_{0,j} \in L_0$, $j \in \{1, \ldots, m\}$. This is a contradiction. ∎

Remarks. 1. The construction in Proposition 2.3 can be used also for infinite cardinal numbers. It suffices to take Y of cardinality α and proceed by transfinite induction up to α. Nevertheless, in the proof of Proposition 2.4 we will show a much simpler construction toward this aim.

2. It is possible to construct a concrete logic $(X', L') \in \mathscr{C}_{n+1} \setminus \mathscr{C}_n$ such that X' is a countable set. It suffices to consider only such sequences in X in the proof of Proposition 2.3 that are constant from some index on and put $L' = \{A \cap X' : A \in L\}$.

Proposition 2.4. For every infinite cardinal number α we have $\mathscr{C}_\alpha \subsetneqq \mathscr{C}_{\alpha^+}$.

Proof. The inclusion $\mathscr{C}_\alpha \subset \mathscr{C}_{\alpha^+}$ is obvious. Let X_1, X_2, X_3, X_4 be disjoint sets each of cardinality α. Set $X = X_1 \cup X_2 \cup X_3 \cup X_4$. Let us define a concrete logic (X, L) in the following way: L consists of the sets $A \subset X$ such that the set $(A \setminus B) \cup (B \setminus A)$ is finite for some set $B \in \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\}$.

Suppose that $A, B \in L$. Since the cardinality of $A \cap B$ is at most α, $A \cap B = \bigcup_{x \in A \cap B} \{x\}$. Since $\{x\} \in L$ for every $x \in X$, we have $(X, L) \in \mathscr{C}_{\alpha^+}$.

On the other hand, every element of L that is a subset of $X_1 = (X_1 \cup X_2) \cap (X_1 \cup X_4)$ is finite. Thus, X_1 cannot be written as a union of less than α elements of L and therefore $(X, L) \notin \mathscr{C}_\alpha$. ■

Theorem 2.5. For every cardinal number α we have $\bigcup_{\beta < \alpha} \mathscr{C}_\beta \subsetneqq \mathscr{C}_\alpha$.

Proof. The inclusion $\bigcup_{\beta < \alpha} \mathscr{C}_\beta \subset \mathscr{C}_\alpha$ is obvious. According to Propositions 2.2–2.4, for every cardinal number β there is a concrete logic $(X_\beta, L_\beta) \in \mathscr{C}_{\beta^+} \setminus \mathscr{C}_\beta$. Let us define the concrete logic (X, L) in the following way [it is in fact the 0–1 pasting of logics (X_β, L_β); see, e.g., Gudder (1979) and Pták and Pulmannová (1991)]:

$$X = \prod_{\beta < \alpha} X_\beta$$

$$L = \left\{ \prod_{\beta < \alpha} A_\beta : A_\beta \in L_\beta \text{ and } A_\beta \neq L_\beta \text{ for at most one } \beta < \alpha \right\}$$

Then it is easy to check that $(X, L) \in \mathscr{C}_\alpha \setminus \bigcup_{\beta < \alpha} \mathscr{C}_\beta$ (all operations are coordinatewise). ■

Proposition 2.6. We have $\bigcup_{\alpha \in \mathrm{card}} \mathscr{C}_\alpha = \mathscr{C}_{\mathrm{card}}$, where card is the class of all cardinal numbers.

Proof. The inclusions $\mathscr{C}_\alpha \subset \mathscr{C}_{\mathrm{card}}$ are obvious. Suppose that

$$(X, L) \in \mathscr{C}_{\mathrm{card}}$$

For every $A, B \in L$ there is a cardinal number $\alpha_{A,B}$ such that $A \cap B$ is the union of less than $\alpha_{A,B}$ elements of L. Then $L \in \mathscr{C}_\alpha$ for $\alpha = \sup\{\alpha_{A,B} : A, B \in L\}$. ■

3. THE CLASSES $\mathscr{C}_\alpha$ AND JAUCH–PIRONNESS

In this section we shall show that there is an interesting link between covering properties (and our classes $\mathscr{C}_\alpha$) and the Jauch–Piron property of states (see, e.g., Jauch, 1968; Piron, 1976; Pták and Pulmannová, 1991). Let us first introduce and recall all properties of states we shall deal with.

Definition 3.1. Let (X, L) be a concrete logic. A *state* on (X, L) is a mapping $s: L \to [0, 1]$ such that:

(1) $s(A \cup B) = s(A) + s(B)$ whenever $A, B \in L$ with $A \cap B = \emptyset$.

(2) $s(X) = 1$ if $X \neq \emptyset$.

A state s is called *Jauch–Piron* if for every $A, B \in L$ with $s(A) = s(B) = 1$ there is a $C \in L$ such that $C \subset A \cap B$ and $s(C) = 1$.

A two-valued state s on (X, L) is said to be *carried by a point* $x \in X$ (and denoted by s_x) if for every $A \in L$ we have $s(A) = 1$ iff $x \in A$.

The set S of (not necessarily all) states on (X, L) is called *full* if for every $A, B \in L$ with $A \not\subset B$ there is a state $s \in S$ such that $s(A) \not\leq s(B)$.

It is easy to see that $s(\emptyset) = 0$ and $s(A^c) = 1 - s(A)$ for every state s on a concrete logic (X, L) and for every $A \in L \setminus \{\emptyset\}$. Let us also observe that the set of all states carried by a point is already full.

A characterization of the class $\mathscr{C}_{\text{card}}$ gives the following proposition.

Proposition 3.2. $\mathscr{C}_{\text{card}}$ is the class of all concrete logics such that every state on it carried by a point is Jauch–Piron. (In particular, every concrete logic of the class $\mathscr{C}_{\text{card}}$ has a full set of two-valued Jauch–Piron states and, on the other hand, every concrete logic with a full set of two-valued Jauch–Piron states has a representation belonging to the class $\mathscr{C}_{\text{card}}$.)

Proof. A concrete logic (X, L) belongs to the class $\mathscr{C}_{\text{card}}$ iff for every pair $A, B \in L$ and for every $x \in A \cap B$ there is a $C \in L$ such that $x \in C \subset A \cap B$. In other words, for every state s_x carried by a point $x \in X$ and for every $A, B \in L$ with $s_x(A) = s_x(B) = 1$ there is a $C \in L$ such that $C \subset A \cap B$ and $s_x(C) = 1$. This proves Proposition 3.2; the remaining part is easy. ■

Proposition 3.3. Every two-valued state on a concrete logic of the class $\mathscr{C}_3$ is Jauch–Piron. On the other hand, there is a concrete logic of the class $\mathscr{C}_3$ with a state that is not Jauch–Piron.

Proof. Suppose that $(X, L) \in \mathscr{C}_3$. Suppose further that s is a two-valued state on (X, L) and $A, B \in L$ with $s(A) = s(B) = 1$. There are $C, D \in L$ such that $A \cap B = C \cup D$. Since the sets $(A \setminus C)$, $(B \setminus D) \in L$ are disjoint, we have either $s(A \setminus C) = 0$ or $s(B \setminus D) = 0$. Thus, either $s(C) = 1$ or $s(D) = 1$. Hence, s is Jauch–Piron.

Let us now take the concrete logic $(X, L) \in \mathscr{C}_3 \setminus \mathscr{C}_2$ of the proof of Proposition 2.2 and a Borel subset B of the interval $[0, 1]$ with a nonrational Lebesgue measure. Then the state s on (X, L) defined, for every $A \in L$, by the formula $s(A) = \lambda(A \cap B)/\lambda(B)$, where λ denotes the Lebesgue measure, is not Jauch–Piron. Indeed, there are $A_1, A_2 \in L$ such that $A_1 \cap A_2 = B$ and for every $A \in L$ with $A \subset B$ we have $s(A) < 1$. ■

Proposition 3.4. Suppose that α is a cardinal number with $\alpha > 3$. Then there is a concrete logic of the class $\mathscr{C}_\alpha$ with a two-valued state that is not Jauch–Piron.

Proof. Let us take the concrete logic (X, L) of the proof of Proposition 2.3 for $n = 3$ and let us define by induction a two-valued state s on (X, L) as follows:

$$s\left(\{a, b\} \times \prod_{i=1}^{\infty} Y\right) = s\left(\{a, d\} \times \prod_{i=1}^{\infty} Y\right) = 1$$

$$s\left(A_k \times \prod_{i=k+1}^{\infty} Y\right) = s\left(P_2(A_k) \times \prod_{i=k}^{\infty} Y\right)$$

for every $k \geq 1$ and for every $A_k \in L_k$.

Then for every $A \in L$ with

$$A \subset \{a\} \times \prod_{i=1}^{\infty} Y = \left(\{a, b\} \times \prod_{i=1}^{\infty} Y\right) \cap \left(\{a, d\} \times \prod_{i=1}^{\infty} Y\right)$$

we have $s(A) = 0$. Hence, s is not Jauch–Piron. ■

Theorem 3.5. The class of concrete logics with the property that every two-valued state is Jauch–Piron is a proper subclass of the class $\mathscr{C}_\omega$, where ω denotes the first infinite cardinal number.

Proof. Suppose that (X, L) is a concrete logic such that every two-valued state on it is Jauch–Piron. Consider the couple (X', L'), where X' is the set of all two-valued states, and A' belongs to L' if and only if there exists $A \in L$ such that A' is exactly the set of all two-valued states s on L with $s(A) = 1$. By applying standard Boolean algebra reasoning, we can prove that L' consist of (not necessarily all) clopen subsets of the compact topological space X' whose base for the open sets is precisely L' [see, e.g., Tkadlec (to appear) for the details]. We can view X as a subset of X' (we adopt the standard identification of the states carried by points of X with the corresponding points of X). Since L' is the base of open sets of X' and since X' is compact, we infer that for every $A', B' \in L'$, the set $A' \cap B'$ is a union of a finite subset of L'. Since $L = \{A \subset X: A = A' \cap X \text{ for some } A' \in L'\}$, we obtain $(X, L) \in \mathscr{C}_\omega$. [It should be noted that an alternative proof of this result can be derived from the technique of the proof of Theorem 3.1 in Navara and Pták (1989).] ■

Theorem 3.6. The class $\mathscr{C}_{\text{card}}$ is a proper subclass of the class of all concrete logics with a full set of two-valued Jauch–Piron states.

Proof. The inclusion follows from Proposition 3.2. Let us take the concrete logic (X, L) of the proof of Proposition 2.4 for some infinite α and define a concrete logic (X', L') as follows:

$$X' = X \cup \{x_1, x_2, x_3, x_4\} \qquad (X \cap \{x_1, x_2, x_3, x_4\} = \emptyset)$$

$$L' = \{A \cup B\colon A \in L \text{ and } B = \{x_i\colon i \in \{1, 2, 3, 4\} \text{ and } A \cap X_i \text{ is infinite}\}\}$$

Then the states carried by points $x \in \{x_1, x_2, x_3, x_4\}$ are not Jauch–Piron and therefore $(X', L') \notin \mathscr{C}_{\text{card}}$. ■

Remarks. (The closedness of $\mathscr{C}_\alpha$ under logic isomorphisms.)

1. The class of concrete logics with a full set of two-valued Jauch–Piron states is the smallest class of concrete logics closed under isomorphisms and containing $\mathscr{C}_{\text{card}}$ (see Proposition 3.2).

2. The class of concrete logics with the property that every two-valued state is Jauch–Piron is the largest class of concrete logics closed under isomorphisms and contained in $\mathscr{C}_{\text{card}}$. [Indeed, every concrete logic (X, L) has a representation (X', L') by means of all two-valued states; $(X', L') \in \mathscr{C}_{\text{card}}$ implies that every two-valued state is Jauch–Piron; see Proposition 3.2.]

3. The classes $\mathscr{C}_0, \mathscr{C}_1, \mathscr{C}_2$ are obviously closed under isomorphisms. On the other hand, according to Proposition 3.4 and Part 2 of this remark, $\mathscr{C}_\alpha$ is not closed under isomorphisms for any $\alpha \geq 4$. It seems to be an open question whether $\mathscr{C}_3$ is closed under isomorphisms.

4. WHEN DOES A CONCRETE LOGIC HAVE TO BE A BOOLEAN ALGEBRA?

In this section we shall discuss the conditions under which a class of concrete logics coincides with the important class $\mathscr{C}_2$ (of concrete Boolean algebras). We improve and extend results of Navara and Pták (1989) in some places.

Let us recall that a subset Y in X, where (X, L) is a concrete logic, is called *dense* in L if for every $A \in L$ there is a $y \in A \cap Y$.

Theorem 4.1. Every concrete logic (X, L) such that each state on it is Jauch–Piron and such that there is a countable dense set Y in X is a Boolean algebra.

Proof. Suppose that $A, B \in L$ with $A \cap B \neq \emptyset$. Then $A \cap B \cap Y$ is a nonempty countable set. Therefore there is a state $s = \sum_{y \in A \cap B \cap Y} a_y s_y$, where

$a_y \in (0, 1)$ are suitable coefficients such that $\sum_{y \in A \cap B \cap Y} a_y = 1$. Since $s(A) = s(B) = 1$ and since s is Jauch–Piron, there is a $C \in L$, $C \subset A \cap B$, such that $s(C) = 1$. Hence, $C \supset A \cap B \cap Y$. Let us suppose that $C \neq A \cap B$. We have $(A \backslash C) \cap (B \backslash C) \neq \emptyset$. Hence, there is a $D \in L$ such that $D \subset (A \backslash C) \cap (B \backslash C)$ and a $y \in D \cap Y \subset (A \cap B \cap Y) \backslash C$. This is a contradiction. ■

In the following proposition we employ a "dimensionlike" notion which might also find an application elsewhere. Suppose that (X, L) is a concrete logic and n is a natural number. We say that *L admits n-dimensional coarsings* if for any pair $A, B \in L$ the following implication holds: If $A \cap B = \bigcup_{i \in I} C_i$, where I is a finite set and $C_i \in L$ for any $i \in I$, then there is a collection $\{D_j : j \in J\}$, where card $J \leq n$ and $D_j \in L$ for any $j \in J$, such that $A \cap B = \bigcup_{j \in J} D_j$ and such that for any C_i $(i \in I)$ there is a $j \in J$ with $C_i \subset D_j$.

Theorem 4.2. Let (X, L) be a concrete logic such that every state on (X, L) is Jauch–Piron. Let us suppose that there is a natural number n such that L admits n-dimensional coarsings. Then L is a Boolean algebra.

Proof. Suppose that a pair $A, B \in L$ is given. We have to show that $A \cap B \in L$. Put $S_{A,B} = \{s: s \text{ is a state on } L \text{ with } s(A) = s(B) = 1\}$. It can be proved by a standard argument that $S_{A,B}$ is a compact set when it is viewed with the pointwise topology [see, e.g., Navara and Pták (1989) for details]. Now, for any $C \in L$ with $C \subset A \cap B$ put $O_C = \{s \in S_{A,B} : s(C) > 1 - 1/n\}$. By the Jauch–Piron property of L, the set $O = \{O_C : C \in L \text{ and } C \subset A \cap B\}$ forms a covering of $S_{A,B}$. Since every set in O is open, the collection O is an open covering of $S_{A,B}$ and we infer, making use of the compactness of $S_{A,B}$, that there is a finite set $\{C_i : i \in I\}$ such that $S_{A,B} = \bigcup_{i \in I} O_{C_i}$. Then $A \cap B = \bigcup_{i \in I} C_i$ and, moreover, for any state $s \in S_{A,B}$ there is an index $i \in I$ such that $s(C_i) > 1 - 1/n$. Let now $\{D_j : j \in J\}$ be an n-dimensional coarsing of $\{C_i : i \in I\}$. If $A \cap B \notin L$, then for any $j \in J$ we can find a point $x_j \in (A \cap B) \backslash D_j$. Let s_j denote the state carried by x_j. Put $s = (1/\text{card } J) \sum_{j \in J} s_j$. Then $s \in S_{A,B}$ but $s(C_i) \leq 1 - 1/n$ for any $i \in I$. This is a contradiction and therefore $A \cap B \in L$. The proof is complete. ■

Corollary 4.3. Let (X, L) be a concrete logic. If every state on (X, L) is Jauch–Piron and L contains only finitely many maximal Boolean subalgebras, then L is a Boolean algebra.

Proof. Let n be the number of all maximal Boolean subalgebras of L. Then one can easily prove that L admits n-dimensional coarsings and this corollary follows from Theorem 4.2. ■

It should be noted that this corollary has been independently obtained in Rogalewicz (1991) as a consequence of deeper results on (generally non-concrete) Jauch–Piron logics.

Let us say that a concrete logic (X, L) is *downward directed* if for every $A, B \in L$ with $A \cap B \neq \emptyset$ there is a $C \in L \backslash \{\emptyset\}$ such that $C \subset A \cap B$. (Let us observe that every concrete logic of the class $\mathscr{C}_{\text{card}}$ is downward directed.)

Proposition 4.4. [See also Navara and Pták (1989) and Tkadlec (1991).] Every downward-directed logic which is a lattice is a Boolean algebra.

Proof. Let (X, L) be a downward-directed logic which is a lattice. Suppose that there are $A, B \in L$ such that $A \wedge B \neq A \cap B$. Since $A \backslash (A \wedge B) \in L$ and $B \backslash (A \wedge B) \in L$ are not disjoint, there is a $C \in L \backslash \{\emptyset\}$ such that

$$C \subset (A \backslash (A \wedge B)) \cap (B \backslash (A \wedge B))$$

Hence,

$$C \cup (A \wedge B) \in L$$

and $A \wedge B \supsetneqq C \cup (A \wedge B) \subset A \cap B$ – a contradiction. ■

Proposition 4.5. Every downward-directed logic (X, L) such that there is no infinite set in L of mutually disjoint elements is a Boolean algebra.

Proof. Suppose that $A, B \in L$ with $A \cap B \neq \emptyset$. Then there is a set $C_1 \in L \backslash \{\emptyset\}$ such that $C_1 \subset A \cap B$. Let us consider sets $(A \backslash C_1)$, $(B \backslash C_1) \in L$. If $(A \backslash C_1) \cap (B \backslash C_1) \neq \emptyset$, then there is a $C_2 \in L \backslash \{\emptyset\}$ such that

$$C_2 \subset (A \backslash C_1) \cap (B \backslash C_1)$$

Proceeding by induction, we obtain a finite set $\{C_1, \ldots, C_n\} \subset L$ of mutually disjoint elements such that $A \cap B = C_1 \cup \cdots \cup C_n \in L$. ■

Proposition 4.6. Every concrete logic (X, L) of the class $\mathscr{C}_3$ that is a σ-logic (i.e., that is closed under countable unions of mutually disjoint elements) is a Boolean algebra.

Proof. Suppose that $A, B \in L$. Let us define by induction $A_k, B_k \in L$ as follows:

$$A_0 = A, \qquad B_0 = B$$

$$A_k, B_k \in L \text{ such that } A_{k-1} \cap B_{k-1} = A_k \cup B_k \text{ for every } k \geq 1$$

Then

$$A \cap B = \bigcup_{k=1}^{\infty} (A_{2k-1} \backslash A_{2k}) \cup \bigcup_{k=1}^{\infty} (B_{2k-1} \backslash B_{2k}) \cup \bigcap_{k=1}^{\infty} A_k \in L$$

because the right side of the equality is a countable union of mutually disjoint elements of L; indeed,

$$\bigcap_{k=1}^{\infty} A_k = \left(\bigcup_{k=1}^{\infty} A_k^c\right)^c = ((A_1^c) \cup (A_2^c \backslash A_1^c) \cup (A_3^c \backslash A_2^c) \cup \cdots)^c \in L$$

This completes the proof. ■

5. OPEN QUESTIONS

Answers to the following questions are presently not known to the authors.

Question 5.1. Is there a concrete logic not belonging to the class $\mathscr{C}_3$ such that every two-valued state on it is Jauch–Piron? (Compare with Proposition 3.3.) If the answer is yes, is the class $\mathscr{C}_3$ closed under isomorphisms?

Question 5.2. Is there a downward-directed logic that does not have a full set of two-valued Jauch–Piron states? (It is easy to see that a concrete logic with a full set of two-valued Jauch–Piron states is downward directed.)

The next question is interesting in connection with the classification presented in Section 2.

Question 5.3. Is it true that every concrete logic with the property that every two-valued state on it is Jauch–Piron belongs to the class $\mathscr{C}_n$ for some natural number $n \geq 4$? (Compare with Proposition 3.4 and Theorem 3.5.)

The last question seems to be of major interest. It has already been posed in Navara and Pták (1989).

Question 5.4. Does every concrete logic each state of which is Jauch–Piron have to be a Boolean algebra? (Compare with Theorem 4.1 and Theorem 4.2.)

It should be noted that in the σ-additive case the answer to this question is no (Bunce *et al.*, 1985).

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